ASYMPTOTIC STABILITY OF A FULL TERM LINEAR DIFFERENCE EQUATION WITH TWO PARAMETERS

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ABSTRACT. The paper introduces an efficient form of necessary and sufficient conditions for a special full term linear difference equation with two real parameters to be asymptotically stable. The result is obtained utilizing the Schur-Cohn criterion. The asymptotic stability region in the parameters plane is also illustrated in the paper.

1. Introduction

The paper deals with asymptotic stability conditions for a full term linear difference equation

\[ y(n + k) + a \sum_{s=1}^{k-1} y(n + s) + by(n) = 0, \quad n = 0, 1, 2, \ldots, \tag{1} \]

where \( a, b \) are nonzero real constants and \( k \geq 2 \) is an integer. The equation (1) arises in the analysis of optimization process of multi-variable quadratic function

\[ f(x_1, \ldots, x_k) = x_1^2 + x_2^2 + \cdots + x_k^2 \]

by the Nelder-Mead method. The convergence of the method is closely related to the asymptotic stability of (1) (see [4]). It is a common knowledge that (1) is asymptotically stable if and only if all roots of its characteristic polynomial

\[ p(\lambda) = \lambda^k + a \sum_{s=1}^{k-1} \lambda^s + b \tag{2} \]

lie inside the unit circle. An assertion connected with this condition introduced in [4] can be reformulated as

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LEMMA 1 (Han, Neumann and Xu [4, Lemma 3.1]). Suppose that \( a, b \in \mathbb{R} \), \( a(b - 1) \neq 0 \), \( |b| \neq 1 \), \( -(b + 1)/a \neq u \), where \( u = -1, 0, 1, 2, \ldots, k - 1 \). Then:

- Assuming that \( a(b - 1) > 0 \) and \( |b| < 1 \) the polynomial (1) has all its roots in the interior of the unit disk if \( -(b + 1)/a > k - 1 \).
- Assuming that \( a(b - 1) < 0 \) and \( |b| < 1 \) the polynomial (1) has all its roots in the interior of the unit disk if \( -(b + 1)/a > -1 \).

The authors of this result developed their proof on the general Schur-Cohn criterion formulated for complex polynomials (see [7]). The aim of the paper is to simplify and improve the conditions stated in Lemma 1. Particularly, we utilize an alternative proof technique to obtain necessary and sufficient conditions for asymptotic stability of (1). This technique was successfully utilized in several special cases of a few term linear difference equations (see [1], [2], [5]). It is surprisingly possible to use it (with a little modification) in the case of the full term equation (1). Moreover, we provide the discussion of asymptotic stability with the construction of stability regions in terms of parameters \( a \) and \( b \).

The structure of the paper is the following: Section 2 introduces some auxiliary terms and it recalls the general form of the Schur-Cohn criterion for real polynomials. Section 3 contains the main result including its proof. The paper is closed by Section 4 giving some remarks in the connection with presented result.

2. Preliminaries

In this section we introduce several terms and we formulate the Schur-Cohn criterion on which the proof of the main result rests.

Let \( M = \Delta_\ell \) be an \( \ell \times \ell \) matrix. We construct \( (\ell - 2) \times (\ell - 2) \) matrix \( \Delta_{\ell-2} \) from \( \Delta_\ell \) by deleting its first and its last column and row. Repeating this procedure we obtain a set of matrices \( \{\Delta_1, \Delta_3, \ldots, \Delta_{\ell-2}\} \) in the case \( \ell \) odd and a set of the matrices \( \{\Delta_2, \Delta_4, \ldots, \Delta_{\ell-2}\} \) in the case \( \ell \) even. The appropriate set of the matrices (with respect to the parity of \( \ell \)) is called the *inners* of the matrix \( M \). Illustrating this we introduce

\[
M_5 = \begin{bmatrix}
m_{11} & m_{12} & m_{13} & m_{14} & m_{15} \\
m_{21} & m_{22} & m_{23} & m_{24} & m_{25} \\
m_{31} & m_{32} & m_{33} & m_{34} & m_{35} \\
m_{41} & m_{42} & m_{43} & m_{44} & m_{45} \\
m_{51} & m_{52} & m_{53} & m_{54} & m_{55}
\end{bmatrix},
\]

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\[
M_6 = \begin{bmatrix}
m_{11} & m_{12} & m_{13} & m_{14} & m_{15} & m_{16} \\
m_{21} & m_{22} & m_{23} & m_{24} & m_{25} & m_{26} \\
m_{31} & m_{32} & m_{33} & m_{34} & m_{35} & m_{36} \\
m_{41} & m_{42} & m_{43} & m_{44} & m_{45} & m_{46} \\
m_{51} & m_{52} & m_{53} & m_{54} & m_{55} & m_{56} \\
m_{61} & m_{62} & m_{63} & m_{64} & m_{65} & m_{66}
\end{bmatrix}.
\]

The inners of matrix \( M_5 \) are \( \Delta_1 \) and \( \Delta_3 \); the inners of matrix \( M_6 \) are \( \Delta_2 \) and \( \Delta_4 \). Much more about inners and their interesting properties and usage can be found in [6]. One useful property of matrices based on inners is the following one: a square matrix \( M \) is said to be positive innerwise if all its inners have positive determinants and \( \det(M) \) is positive too.

Now we introduce a general form of determinantal criterion, which enables us to decide whether or not all roots of polynomial

\[
P(\lambda) = \lambda^k + a_{k-1}\lambda^{k-1} + \cdots + a_1\lambda + a_0,
\]

where \( a_i, i = 0, \ldots, k-1 \) are reals, lie inside the unit circle in the complex plane.

**Proposition 1** (The Schur-Cohn criterion [3, Theorem 5.1]). The polynomial (3) has all its roots inside the unit circle if and only if it holds that:

(i) \( P(1) > 0 \);
(ii) \( (-1)^kP(-1) > 0 \);
(iii) \((k-1) \times (k-1)\) matrices

\[
B_{k-1} = \begin{bmatrix}
1 & 0 & \cdots & 0 \\
a_{k-1} & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots \\
a_2 & \cdots & a_{k-1} & 1
\end{bmatrix} \pm \begin{bmatrix}
0 & \cdots & 0 & a_0 \\
0 & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots \\
a_0 & a_1 & \cdots & a_{k-2}
\end{bmatrix}
\]

are positive innerwise.

3. Main result

We introduce the necessary and sufficient conditions for difference equation (1) to be asymptotically stable.

**Theorem 1.** Let \( a \) and \( b \) be real nonzero constants. If \( k \geq 2 \), then (1) is asymptotically stable if and only if

\[
b > a - 1, \quad b < 1, \quad b > (1-k)a - 1.
\]
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Proof. The proof is based on the determinantal criterion recalled in Proposition 1 applied to (2), which is the characteristic polynomial of (1). We investigate the odd and the even case of \( k \) apart.

3.1. The case of \( k \) odd

The conditions (i)–(iii) of Proposition 1 applied to (2) with odd \( k \) turn to:

(i)* \( 1 + (k-1)a + b > 0; \)

(ii)* \( 1 - b > 0; \)

(iii)* \( (k-1) \times (k-1) \) matrices \( B^+_{k-1} \) and \( B^-_{k-1} \) given by

\[
B^\pm_{k-1} = \begin{bmatrix}
1 & 0 & 0 & 0 & \cdots \\
0 & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots \\
0 & \cdots & \ddots & \ddots \\
a & \cdots & \ddots & a & 1
\end{bmatrix} \pm \begin{bmatrix}
0 & 0 & b & \cdots \\
0 & \ddots & \ddots & \cdots \\
\vdots & \ddots & \ddots & \cdots \\
0 & \cdots & \ddots & a \\
b & \cdots & \ddots & a
\end{bmatrix}
\] (6)

are positive innerwise.

It remains to simplify the third condition. First we do the following row and column additions within \( B^+_{k-1} \):

• We add \((-1)\)-multiple of the \((k-i)\)th column to the \(i\)th column, \( i = 1, 2, \ldots, (k-1)/2 \).

• We add the \(i\)th row to the \((k-i)\)th row, \( i = 1, 2, \ldots, (k-1)/2 \).

Now we realize the following row and column additions within \( B^-_{k-1} \):

• We add the \((k-i)\)th column to the \(i\)th column, \( i = 1, 2, \ldots, (k-1)/2 \).

• We add the \((-1)\)-multiple of the \(i\)th row to the \((k-i)\)th row, \( i = 1, 2, \ldots, (k-1)/2 \).

By the above mentioned operations, which preserve the determinants of the inners, we obtain

\[
\hat{B}^+_{k-1} = \begin{bmatrix}
C & D \\
O & F
\end{bmatrix}, \quad \hat{B}^-_{k-1} = \begin{bmatrix}
C & -D \\
O & G
\end{bmatrix},
\]

where \( C, D, O, F, G \) are \(((k-1)/2) \times ((k-1)/2)\) matrices

\[
C = \begin{bmatrix}
1 - b & 0 & 0 & \cdots \\
0 & \ddots & \ddots & \cdots \\
\vdots & \ddots & \ddots & \cdots \\
0 & \cdots & \ddots & 0 \\
0 & \cdots & 0 & 1 - b
\end{bmatrix}, \quad D = \begin{bmatrix}
0 & 0 & b & \cdots \\
0 & \ddots & \ddots & \cdots \\
\vdots & \ddots & \ddots & \cdots \\
0 & \cdots & \ddots & a \\
b & \cdots & \ddots & a
\end{bmatrix},
\] (7)
and $O$ is the zero matrix. Then we can capture the determinants of the inners and matrices $\hat{B}_{k-1}^+ = \Delta_{k-1}^+$, $\hat{B}_{k-1}^- = \Delta_{k-1}^-$ as:

$$\det \Delta_m^+ = (1 - b)^{m/2}(1 - a + b)^{m/2 - 1}(1 + (m - 1)a + b), \quad m = 2, 4, 6, \ldots, k - 1;$$

$$\det \Delta_m^- = (1 - b)^{m/2}(1 - a + b)^{m/2}, \quad m = 2, 4, 6, \ldots, k - 1.$$

Considering (ii*) and $\det \Delta_m^- > 0$ for $m = 2, 4, 6, \ldots, k - 1$ we obtain $1 - a + b > 0$. Hence conditions (i*)–(iii*) imply (5). The reverse implication is obvious with respect to the fact that assuming $b > a - 1$ the relation $1 + (k - 1)a + b > 0$ implies $1 + (m - 1)a + b > 0$ for all $m = 2, 4, 6, \ldots, k - 1$. The case of $k$ odd is proved.

### 3.2. The case of $k$ even

The conditions (i)–(iii) of Proposition 1 applied to (2) with even $k$ turn to:

- (i**) $1 + (k - 1)a + b > 0$;
- (ii**) $1 - a + b > 0$;
- (iii**) $(k - 1) \times (k - 1)$ matrices $B_{k-1}^+$ and $B_{k-1}^-$ given by (6) are positive innerwise.

We again need to simplify the third condition. We do the next row and column additions within $B_{k-1}^+$:

- We add $(-1)$-multiple of the $(k - i)$th column to the $i$th column, $i = 1, 2, \ldots, (k - 2)/2$.
- We add the $i$th row to the $(k - i)$th row, $i = 1, 2, \ldots, (k - 2)/2$.

We do the next row and column additions within $B_{k-1}^-$:

- We add the $(k - i)$th column to the $i$th column, $i = 1, 2, \ldots, (k - 2)/2$.
- We add the $(-1)$-multiple of the $i$th row to the $(k - i)$th row, $i = 1, 2, \ldots, (k - 2)/2$. 

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By the above mentioned operations, which preserve the determinants of the
inners and the determinants of the matrices, we get
\[
\hat{B}_{k-1}^+ = \begin{bmatrix}
C & 0 & D \\
0 & 1 + b & a - a \\
O & 2a & F
\end{bmatrix}, \quad \hat{B}_{k-1}^- = \begin{bmatrix}
C & 0 & -D \\
0 & 1 - b & -a - a \\
O & 0 & G
\end{bmatrix},
\]
where \( C, D, O, F, G \) are \( ((k - 2)/2) \times ((k - 2)/2) \) matrices given by the
schemes (7), (8) and \( O \) is the zero matrix.

Investigating the determinants of the inners of \( B_{k-1}^+ \) we first factor out \( 1/2 \)
from the middle row. Then we can utilize the relation known for \((\ell \times \ell)\)
real matrix
\[
\det \begin{bmatrix}
r + q_1 & r & \cdots & r \\
r & r + q_2 & \cdots & r \\
r & r & \cdots & r + q_\ell
\end{bmatrix} = q_1 q_2 \cdots q_\ell \left(1 + r \sum_{i=1}^{\ell} 1/q_i\right),
\]
because such submatrix appears in the each inner in its right bottom part.
Indeed, we obtain such submatrix by a substitution \( r = 2a, q_1 = 2(1 - a + b) \)
and \( q_i = 1 - a + b, i = 2, 3, \ldots, \ell. \)

Then we can express determinants of the inners and matrices
\[
\hat{B}_{k-1}^+ = \Delta_{k-1}^+, \quad \hat{B}_{k-1}^- = \Delta_{k-1}^-
\]
as
\[
\det \Delta_m^+ = [(1 - b)(1 - a + b)]^{(m-1)/2} (1 + (m - 1)a + b), \quad m = 1, 3, 5, \ldots, k - 1;
\]
\[
\det \Delta_m^- = (1 - b)^{(m+1)/2} (1 - a + b)^{(m-1)/2}, \quad m = 1, 3, 5, \ldots, k - 1.
\]

Considering \((ii^*)\) and \( \det \Delta_m^- > 0 \) for \( m = 1, 3, 5, \ldots, k - 1 \) we obtain \( 1 - b > 0. \)
The rest of the proof is analogous to the case of odd \( k. \) Hence the case of \( k \) even
is proved. \( \square \)

4. Comments and conclusions

We conclude the paper by several remarks and discussion of geometrical as-
pects of the problem.

First we compare the main result with Lemma \( \Box \) We emphasize that The-
orem \( \Box \) gives necessary and sufficient condition for asymptotic stability of \( \Box \)
in a compact form, whereas Lemma \( \Box \) is formulated just as a sufficient condition.
Moreover, Theorem 1 is free of the additional assumptions on the coefficients $a, b$ (i.e., $a(b - 1) \neq 0$ and $-(b + 1)/a \neq u$, $u = -1, 0, 1, \ldots, k - 1$).

These assumptions result from another proof technique utilized in [4], which is based on the Schur-Cohn criterion in a more general form. Particularly, the number of sign changes in a specific nonzero sequence of certain determinants is investigated, which corresponds to the number of zeros of complex polynomial inside the unit disk. The extra assumptions in Lemma 1 guarantee nonzero values of these determinants. The cases, when any of the determinants has zero value, must be investigated separately. Such singular cases do not occur in our proof technique.

Now we introduce several comments to the geometrical interpretation of the main result. The asymptotic stability region in the plane $(a, b)$ for the equation (1) is a triangle with vertices $(0, -1), (2, 1), (2/(1 - k), 1)$ (see Fig. 1). The higher $k$ we consider the smaller the region is (the closer is the last mentioned vertex to the point $(0, 1)$). It also coincides with the so called delay dependent asymptotic stability region and delay independent stability region. The area in the plane $(a, b)$ constrained by $b < 1$, $a \geq 0$, $b > a - 1$ is the delay independent asymptotic stability region. Taking into account any pair of the parameters $(a, b)$ from this region we obtain the asymptotically stable difference equation (1) considering any integer $k \geq 2$.

![Figure 1. Asymptotic stability region of (1) in $(a, b)$ plane.](image)

On the contrary, if we consider a pair $(a, b)$, $a < 0$ from the asymptotic stability region of (1) for a fixed $k = k^*$, then there exists integer $k^{**} > k^*$ such that this pair $(a, b)$ is outside of the asymptotic stability region of (1) considering any $k \geq k^{**}$. The delay dependent asymptotic stability region is given by the relations $b < 1$, $a < 0$ and $b > (1 - k)a - 1$. 

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We now compare the result with a well known sufficient condition for asymptotic stability of (1), namely the Cohn stability domain, which can be expressed as

\[ |b| + (k-1)|a| < 1. \]

The region given by this relation is a diamond in the plane \((a, b)\) with vertices \((0, 1), \left(-1/(k-1), 0\right), (0, -1), \left(1/(k-1), 0\right)\). It is naturally a subset of the asymptotic stability region of (1). Notice that one of the diamond’s sides is a part of the stability triangle’s side given by \(b = (1-k)a - 1\). The higher \(k\) we consider the lower value the ratio of areas of the “stability diamond” to the “stability triangle” has.

The author believes that the proof technique utilized in the paper can be applied to some special cases of a three-parametric full term difference equation. It remains an open problem to formulate some efficient form of necessary and sufficient conditions of asymptotic stability for such equations.

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